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Matrix solutions of wave equations and Clifford algebras

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Abstract. We are extending the formation of matrix solutions u_n for linear and nonlinear wave equations by construction of unitary anti-Hermitian–anti-commuting matrices up to the eighth order. We use Clifford algebras $C(0, n)$ with periodicity in modulo 8 to construct coupled matrix solutions. We also propose to use the matrix solutions for describing the intrinsic rotations of particles.

The proposed formation of the matrix solutions for wave equations includes both passage to the moving frame of reference, reducing wave equations to ordinary differential equations, and construction of the unitary anti-Hermitian–anti-commuting matrices of the nth order. The conventional matrix algebra as in [1–3] is complemented by the Clifford algebra $C(0, n)$ [4–6], which allows us, due to the periodicity $C(0, n)$ in modulo 8, to obtain not only the whole family of matrix solutions u_n for n up to 8 but also coupled solutions. The unified exponential form of matrix solutions u_n makes it possible to obtain various rotational properties of these solutions. In the case of the linear wave equation for a free particle the matrix solutions with their rotational properties are proposed for description of intrinsic rotations of the particle.

Our basic construction is a family of matrix functions

$$u_n(\phi) = \cos(\phi)E_n + \sin(\phi)\sum_{j=1}^{m} a_j M_j \qquad \sum_{j=1}^{m} a_j^2 = 1 \qquad n = 1, 2, \ldots \tag{1}$$

where E_n is the unit $(n \times n)$-matrix. Complex linear-independent $(n \times n)$-matrices $M_j (j = 1, 2, \ldots, m)$ should possess the following properties: they are unitary ($M_j^* = M_j^{-1}$), anti-Hermitian ($M_j^* = -M_j$) and anti-commuting ($M_i M_j = -M_j M_i$). The symbol $*$ denotes the transition to a complex conjugate transposed matrix. For $n = 1$ we should set $m = 1$ and $M_1 = \mathrm{i}$ (i—imaginary unit). It is known [1–3] that $m = 3$ for $n = 2$, $m = 1$ for $n = 3$, $m = 4$ for $n = 4$. The angular parameter ϕ is taken in two forms: $\phi = kz$ in the linear case and $\phi = \phi(\alpha z)$ in the nonlinear case, where parameters k and α and variable z will be defined below.

If a certain matrix M is unitary and anti-Hermitian then $M^{-1} = -M$. On the other hand if any $(n \times n)$-matrix M satisfies the property $M^2 = -E_n$ then $M = -M^{-1}$. Thus we can state that the algebra of matrices M_j is based on two main properties:

$$M_j^2 = -E_n \qquad \text{and} \qquad M_i M_j = -M_j M_i \qquad \text{for} \quad i \neq j. \tag{2}$$

Properties (2) allow us to represent function u_n in the form of a matrix exponent

$$u_n(\phi) = \exp(\mathrm{i}\phi a_j A_j) \qquad \text{where} \quad A_j = -\mathrm{i}M_j. \tag{3}$$

Product $a_j A_j$ means summing over $j = 1, \dots, m$; matrices A_j are Hermitian in contrast to anti-Hermitian matrices M_j.

Let us start with construction of matrices M_j which satisfy (2). For $n = 2$ we form a matrix function u_2 via unit quaternions, i.e. we take $M_j = H_j$ where

$$H_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad H_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

For $n = 4$ the following matrices can be proposed:

$$B_j = \begin{pmatrix} H_j & 0 \\ 0 & -H_j \end{pmatrix} \qquad \text{for} \quad j = 1, 2, 3 \qquad \text{and} \qquad B_4 = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}.$$

For $n = 8$ we form the matrices

$$C_j = \begin{pmatrix} B_j & 0 \\ 0 & -B_j \end{pmatrix} \qquad \text{for} \quad j = 1, 2, 3, 4 \qquad \text{and} \qquad C_5 = \begin{pmatrix} 0 & E_4 \\ -E_4 & 0 \end{pmatrix}.$$

It is easy to verify that the four matrices B and the five matrices C are linear-independent and satisfy (2).

Another way of constructing matrices of the fourth and the eighth order is based on application of the Pauli and the Dirac matrices. Note that the Pauli matrices σ_j are Hermitian and connected with quaternions via relations

$$\sigma_1 = -\mathrm{i}H_1 \qquad \sigma_2 = \mathrm{i}H_2 \qquad \sigma_3 = -\mathrm{i}H_3.$$

The Dirac (4×4)-matrices γ_j for $j = 1, 2, 3, 4$ are formed from σ-matrices as usual

$$\gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \qquad \text{for} \quad j = 1, 2, 3 \qquad \text{and} \qquad \gamma_4 = \begin{pmatrix} \mathrm{i}E_2 & 0 \\ 0 & -\mathrm{i}E_2 \end{pmatrix}.$$

In an analogous way we can form (8×8)-matrices from Hermitian $\mathrm{i}\gamma_j$-matrices, i.e. we set

$$D_j = \begin{pmatrix} 0 & \mathrm{i}\gamma_j \\ -\mathrm{i}\gamma_j & 0 \end{pmatrix} \qquad \text{for} \quad j = 1, 2, 3, 4 \qquad \text{and} \qquad D_5 = \begin{pmatrix} \mathrm{i}E_4 & 0 \\ 0 & -\mathrm{i}E_4 \end{pmatrix}.$$

We define more exactly the two triples of (6×6)-matrices F from [3] as

$$F_j = \begin{pmatrix} H_j & 0 \\ 0 & \gamma_j \end{pmatrix} \qquad \text{and} \qquad F_{j+3} = \begin{pmatrix} H_j & 0 \\ 0 & \gamma_{j+1} \end{pmatrix} \qquad \text{for} \quad j = 1, 2, 3.$$

Each triple satisfies (2) and all six matrices are linear-independent.

For odd n it is known [3] that there do not exist two matrices which anti-commute with each other in a set of unitary anti-Hermitian (3×3)-matrices. However, as is shown in [1] solution u_3 with a single unitary anti-Hermitian matrix has a rich intrinsic structure in a subspace based on Hermitian matrices. The Clifford algebra $C(0, n)$ for odd n gives additional possibilities for construction of the matrix solution. The algebra $C(0, n)$ has n generators e_j which obey properties (2) for $j = 1, \dots, n$. This algebra is realized as that of pairs $[a, b]$, $[c, d]$, ... with operations of addition ($[a + c, b + d]$), multiplication ($[ac, bd]$) and multiplication by real numbers ($[\lambda a, \lambda b]$).

It follows from the Salingaros result [5] that $C(0, 3)$ is realized in the form of the direct sum $H \oplus H$ where H are the quaternions H_j. It is easy to verify that element $\hat{M}_j = [H_j, H_j]$ satisfies the properties (2) provided $\hat{E}_n = [E_n, E_n]$ is the unit in $C(0, n)$. Thus we construct, in addition to u_3, a matrix function in the form

$$u_3^+(\phi) = \cos(\phi)[E_2, E_2] + \sin(\phi) \sum_{j=1}^{3} a_j [H_j, H_j].$$

Due to operations of addition and multiplication by real numbers the function u_3^+ can be written in the form $u_3^+ = [u_2, u_2]$. Note that the pair $[H_j, H_j^*]$ can be used to form the function $u_3^- = [u_2, u_2^*]$. Analogously, we can form the matrix functions of the fifth order $u_5^+ = [u_4, u_4]$ and $u_5^- = [u_4, u_4^*]$ and of the seventh order $u_7^+ = [u_6, u_6]$ and $u_7^- = [u_6, u_6^*]$. These two forms, u^+ and u^-, of matrix solutions can be useful for description of coupled particles and particle–anti-particle pairs.

We restrict ourselves by constructing unitary anti-Hermitian anti-commuting matrices up to the eighth order because of the Coquereaux theorem of periodicity [6], which states that all algebras $C(0, n+8)$ are isomorphic to the direct product of $C(0, n)$ and $C(0, 8)$.

Consider now the nonlinear Klein–Gordon (KG) equation for normalized value u in natural units $c = 1 = \hbar$

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{\mathrm{d}Q}{\mathrm{d}u} = 0 \qquad Q(u) = \frac{\lambda^2}{4}(u^2 - 1)^2. \tag{4}$$

It follows from the form of the anharmonic potential Q with two minima (basic states $u = 1$ and -1) that equation (4) describes a transition from one basic state to another. The parameter λ is the mass. According to our approach we choose in the three-dimensional space the direction $x = \sum_{j=1}^3 c_j x_j$ with $\sum_{j=1}^3 c_j^2 = 1$, and next we pass to the moving frame of reference $z = x - vt$ (here v is the velocity) and reduce equation (1) to the ordinary differential equation

$$(v^2 - 1)\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = -\lambda^2(u^3 - u). \tag{5}$$

Let us prove that all forms of the matrix functions constructed above satisfy the KG equation for a special choice of the argument ϕ.

Theorem 1. *The matrix functions u_n, u_n^*, u_j^+, u_j^- for indices $n = 1, 2, \ldots, 8$ and $j = 3, 5, 7$ and for argument*

$$\phi \equiv \phi(\alpha z) = \operatorname{arccot}(-\sinh(\alpha z)) \qquad \alpha = \lambda\sqrt{\frac{2}{1 - v^2}} \qquad v^2 < 1$$

satisfy equation (5) and, hence, (4).

Proof. Consider the function $u_n(\phi)$. From the definition of the angular parameter we find $\cos(\phi) = -\tanh(\alpha z)$ and $\sin(\phi) = \operatorname{sech}(\alpha z)$. Differentiating the last equality with respect to z and taking into account the preceding equality we obtain $\phi' = \alpha \sin(\phi)$ and $\phi'' = \alpha^2 \sin(\phi)\cos(\phi)$. To simplify the expressions we rewrite (1) in the form $u = \cos(\phi) + \sin(\phi)\Sigma$ where E_n is replaced by unity and the matrix sum is replaced by Σ. It follows from (2) and the unit norm of vector a that equality $\Sigma^2 = -1$ is valid. Simple computations lead to the following two equalities:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \alpha^2 \sin(\phi)(-\sin(2\phi) + \cos(2\phi)\Sigma)$$
$$u(u^2 - 1) = 2\sin(\phi)(-\sin(2\phi) + \cos(2\phi)\Sigma).$$

Now, due to the relation $\alpha^2(v^2 - 1) = -2\lambda^2$ we state that the function $u = u_n(\phi)$ satisfies equation (5). In turn, for Hermitian conjugated matrix function u_n^* it is clear that $M_j^* = -M_j$ satisfies (2) and, hence, $\Sigma^2 = -1$. Thus in the same way as above it is proved that u_n^* also satisfies (5). Now it is evident that functions u_j^+ and u_j^- satisfy (5) as well. The theorem is proved. □

Note that the solution $u_1(\phi) = \exp(\mathrm{i}\phi)$ varies from 1 to -1 and its negative double $-u_1(\phi) = \exp(\mathrm{i}(\phi + \pi))$ varies from -1 to 1 if ϕ increases from 0 (for $z = -\infty$) to π (for

$z = \infty$). The solution $u_2(\phi) = \exp(\mathrm{i}\phi a_j A_j)$ (summing over $j = 1, 2, 3$) for $A_j = -\mathrm{i}H_j$ varies from E_2 to $-E_2$ in the same region of ϕ.

Consider now a linear wave equation, the so-called Klein–Gordon–Fock (KGF) equation [4, 7]

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \mu^2 u = 0 \tag{6}$$

where μ is the mass. Equation (6) can be derived from (4) if one takes $Q = \mu^2 u^2/2$. For a free particle equation (6) for $c = 1 = \hbar$ is obtained from the relativistic relation $E^2 = p^2c^2 + \mu^2c^4$ [7] by using operators of energy E and momentum $p = (p_1, p_2, p_3)$ in the form, respectively, $\mathrm{i}\hbar\partial/\partial t$ and $p_j = -\mathrm{i}\hbar\partial/\partial x_j$. Note that the Schrödinger equation $\mathrm{i}\partial\psi/\partial t = H\psi$ for Hamiltonian $H = \sqrt{p^2 + \mu^2}$, when squared, also gives the same equation (6).

If the origin of the moving frame of reference is placed at a pointwise particle so that $p = 0$, then equation (6) turns into the equation of a harmonic oscillator

$$\frac{\partial^2 u}{\partial t^2} + \omega^2 u = 0 \qquad \text{where} \quad \omega^2 = \mu^2. \tag{7}$$

For $u_n(\phi)$ in the exponential form (3) with $\phi = kz$ we find due to $(a_j A_j)^2 = E_n$ that $\partial^2 u_n/\partial t^2 = -v^2k^2u_n$. Thus $u_n(kz) = \exp(\mathrm{i}kza_jA_j)$ satisfies (7) and, hence, (6) provided $k^2v^2 = \omega^2 = \mu^2$.

Each unitary matrix $u_n(\phi)$ ($n = 2, \ldots, 8$) performs a certain rotation. For example, spin-matrix $u_2(\phi)$ according to the terminology of Penrose and Rindler [8] turns a unit sphere by an angle 2ϕ around vector $a = (a_1, a_2, a_3)$. It is natural to connect the direction of vector a with that of the intrinsic angular momentum, i.e. with the spin vector. The matrix u_3 describes rotations around three directions corresponding to basis vectors in a space of Hermitian matrices A_j [1].

Due to the fact that $u_n(kz + C)$ also satisfies (7) and (6) for any real C we can construct a geometrical image of the pair $u_1(C)$ and $u_2(C)$ in the form of a torus for $0 \leqslant C \leqslant 2\pi$ and for $z = 0$. Indeed, the solution $u_1(C)$ describes the unit cycle. Let the point $\exp(\mathrm{i}C)$ for fixed C be the centre of the unit sphere. In the time in which the parameter C varies from 0 to 2π the solution $u_2(C)$ turns the unit sphere in such a way that the sphere forms a torus and each point on the sphere describes a wound curve around the unit cycle.

The matrix $u_4(kz) = \exp(\mathrm{i}kza_jA_j)$ for $A_j = -\mathrm{i}\gamma_j$ is remarkable in the fact that it satisfies the Dirac equation. Consider, indeed, the Dirac equation in the form [9]

$$(\mathrm{i}\gamma_0\partial/\partial t - \mathrm{i}\gamma_j p_j - \mu)u = 0 \qquad \text{where} \quad \mathrm{i}\gamma_0 = \gamma_4. \tag{8}$$

The matrix $u_4(kz)$ being substituted into (8), provided $p = 0$ is in the moving frame of reference, gives $-kv\gamma_4 a_j\gamma_j - \mu = 0$. This equation is satisfied for $kv = \mu$, $a_4 = 1$ and $a_1 = a_2 = a_3 = 0$ due to the property $\gamma_j^2 = -1$. It is evident that $u_4^*(kz)$ satisfies the Hermitian-conjugated Dirac equation. The KGF equation (6) can be decomposed via (8×8)-matrices D_j into two equations: a basic one and another Hermitian-conjugated one (analogously to the Dirac equations) with solutions u_8 and u_8^* which are based on matrices D_j and D_j^*, respectively.

The solutions u_4 and u_8 constructed by use of matrices B_j and C_j, respectively, have another significant property: $u_4 = \mathrm{diag}(u_2, u_2^*)$ for $a = (a_1, a_2, a_3, 0)$ and $u_8 = \mathrm{diag}(u_4, u_4^*)$ for $a = (a_1, a_2, a_3, 0, 0)$. Due to this diagonal representation in the same way as for spin-matrix $u_2(\phi)$ the following result can be derived: the matrix solution $u_4(\phi)$ performs a rotation by the angle 4ϕ and the matrix solution $u_8(\phi)$ performs a rotation by the angle 8π. We hope that matrix solutions u_n with their rotational properties will be useful for description of the intrinsic rotations of the particles.

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